Lecture 21
15.2 Double and iterated integrals over general regions
15.3 Area

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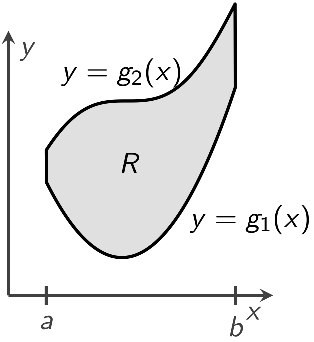
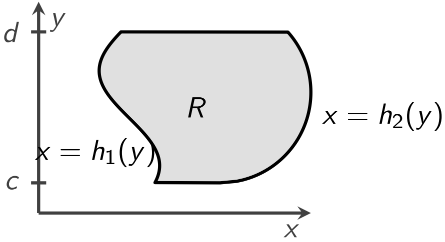
Last class



$$R : g_1(x) \leq y \leq g_2(x)$$
$$a \leq x \leq b$$



$$R : h_1(y) \leq x \leq h_2(y)$$
$$c \leq y \leq d$$

Fubini's Theorem (for a general region)

Theorem

If a region R is given by the inequalities
$R : g_1(x) \leq y \leq g_2(x), \quad a \leq x \leq b$, *then the signed volume below* $f(x, y)$ *and above R is given by*

$$\int\int_R f(x,y)dA = \int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)dy\ dx$$

and conversely if R is given by the inequalities
$R : h_1(y) \leq x \leq h_2(y), \quad c \leq y \leq d$ *then the signed volume below* $f(x, y)$ *and above R is given by*

$$\int\int_R f(x,y)dA = \int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)dx\ dy$$

Example

Example

Let R be the region bounded by the curve $x = y^2$, $y = 1$, *and* $x = 0$. *Find the integral of* $f(x, y) = 3y^3 e^{xy}$ *over R.*

Example

Example

Let R be the region bounded by the curve $x = y^2$*,* $y = 1$*, and* $x = 0$*. Find the integral of* $f(x, y) = 3y^3 e^{xy}$ *over R.*

We can describe R with the bounds $0 \leq x \leq y^2, \quad 0 \leq y \leq 1$, or with the bounds $\sqrt{x} \leq y \leq 1, \quad 0 \leq x \leq 1$.

Example

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Let R be the region bounded by the curve $x = y^2$, $y = 1$, and $x = 0$. Find the integral of $f(x, y) = 3y^3 e^{xy}$ over R.

We can describe R with the bounds $0 \leq x \leq y^2, \quad 0 \leq y \leq 1$, or with the bounds $\sqrt{x} \leq y \leq 1, \quad 0 \leq x \leq 1$.

Which integral will be easier to calculate?

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We can describe R with the bounds $0 \leq x \leq y^2, \quad 0 \leq y \leq 1$, or with the bounds $\sqrt{x} \leq y \leq 1, \quad 0 \leq x \leq 1$.
Which integral will be easier to calculate?
We put the x-bounds on the inside and integrate.

$$\int_{y=0}^{y=1} \int_{x=0}^{x=y^2} 3y^3 e^{xy} dx\ dy = \int_{y=0}^{y=1} \frac{3y^3 e^{xy}}{y} \Bigg]_{x=0}^{x=y^2} dy$$

$$= \int_{y=0}^{y=1} 3y^2 e^{y^3} - 3y^2 dy = e^{y^3} - y^3 \Bigg]_{y=0}^{y=1} = e - 1 - (1 - 0) = e - 2$$

Example

Example

Evaluate the following integral:

$$\int_{x=0}^{x=\pi} \int_{y=x}^{y=\pi} \frac{\sin(y)}{y} dy \ dx$$

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The integral on the inside doesn't have an antiderivative that can be expressed in terms of elementary functions.

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So use Fubini's Theorem!

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The integral on the inside doesn't have an antiderivative that can be expressed in terms of elementary functions.
So use Fubini's Theorem!

$$\int_{x=0}^{x=\pi} \int_{y=x}^{y=\pi} \frac{\sin(y)}{y} dy \ dx = \int_{y=0}^{y=\pi} \int_{x=0}^{x=y} \frac{\sin(y)}{y} dx \ dy$$

$$= \int_{y=0}^{y=\pi} \sin(y) dy = \cos(y) \Big]_{y=0}^{y=\pi} = 1 + 1 = 2.$$

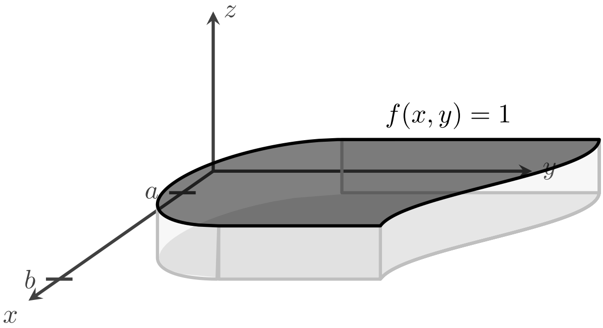
15.3 Area

Definition

The area of a closed, bounded region R is

$$A = \int\int_R dA$$

(i.e., an integral over R with $f(x, y) = 1$ on the inside).



Area example

Example

Find the area of the region R bounded by $y = x$ *and* $y = x^2$ *in the first quadrant.*

Area example

Example

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The area is

$$\int_{x=0}^{x=1} \int_{y=x^2}^{y=x} (1) dy \ dx = \int_{x=0}^{x=1} (x - x^2) dx =$$

$$\left. \frac{x^2}{2} - \frac{x^3}{3} \right]_{x=0}^{x=1} = 1/2 - 1/3 = 1/6$$